Alphamagic Squares, Part II

More adventures with abacus and alphabet, extending explorations into the untrodden realms of computational logology.

by Lee C. F. Sallows

Part I of this article (in the Fall 1986 ABACUS*) described how the recent translation of a rediscovered fifth-century Runic inscription—the* Li shu, *long mislaid in the vaults of the British Library—led to computer investigation of a fascinating, novel structure, the* alphamagic square *(see Figure 1). In an ordinary magic square, such as the ancient Chinese* Lo shu, *distinct numbers are cunningly arranged so that their row, column, and diagonal sums turn out to be identical. Such a square would be* alphamagic *if, when you replace the usual numerals with their number-names written out in English (or some other language), each row, column, and diagonal is seen to contain an equal number of* letters *as well. The letter-count of a number's written name is known as its "logorithm."*

Part I described an algorithm for detecting all possible 3×3 alphamagics. The results of investigations in some thirty different languages were presented and many curiosities unearthed, including special cases where a square in one tongue remains alphamagic when translated into a second language. Here in Part II, the author pushes on to examine the higher orders—squares larger than 3×3—revealing a large area yet to be explored, with richer challenges for recreationally oriented programmers.

Higher Orders

Readers who may be regretting that most of the really worthwhile nuggets have already been culled from the alphamagical goldfield are in for a pleasant surprise. With the transition from order 3 to order 4, and higher, comes a concomitant jump in the perplexities confronting our advance, since hindsight reveals order 3 as a special, unusually tractable case. The problems involved having largely resisted solution thus far, this higher ground has been barely surveyed, let alone exhausted. As a result, it is no exaggeration to say that for programmers and pencil-owners alike, there remain rich pickings to be had, given ingenuity and the will to explore. Before turning to the difficulties imposed, however, it will be well to distinguish between logologist's gold and fool's gold.

In the previous issue we looked at greco-latin squares, noting obvious isomorphisms between the single instance of order 3 and certain 3×3 alphamagics. [A *latin* square of order N, it will be recalled, is defined as one comprising N^2 entries of N distinct elements, each occurring exactly once in every row and column. *Greco-latins* are formed when two suitable latin squares are appended so that all the resultant compound entries are unique. Only one square of 3×3 exists.] For all higher orders, however, assorted kinds of greco-latin squares exist—in particular, those using latin squares in which the N distinct elements also appear along both diagonals; see Figure 2.

Now an interesting if obvious property of these matrices is that their elements are always replaceable by appropriate numbers so as to produce a nontrivial magic square. In Figure 3, for example, A=20, B=30, C=40, D=50, a=6, b=7, c=8, d=9, and the magic constant is 170. A "*diagonal* greco-latin square," in other words, is a recipe for certain types of magic square. Less prominent perhaps, but equally true, is that it yields a recipe for certain types of alphamagic square too.

There is a neat trick that can be

used for trapping friends into scornful expressions of baseless incredulity. You show someone the *Li shu* and explain its properties. While your subject is still goggling under its impact, mention casually that this is only kid's stuff; you yourself have produced an alphamagic cube of order 8. "An 8×8×8 alphamagic *cube* using entirely different number-words in every single position?" comes the unbelieving response. "Sure," you reply. "Not only that, the rows, columns, pillars and diagonals are all perfect *anagrams* of each other." Keeping a straight face at this point, be prepared to return any searching glances. Eventually your victim will be forced into a demurral. "But that's nothing," you retort, "my cube even retains all its alphamagic properties when translated into French. . . ." Puzzlement mixed with skepticism will now spill over into indignation at the leg-pulling. It is time suavely to produce your piece of paper showing the three superimposed order-8 latin cubes.

Order-8 greco-latin cubes, in fact, offer little difficulty in construction; for example, the interested reader can consult *Latin Squares and Their Applications* by Denes and Keedwell. Constructions of this size being cu(m)bersome, however, let us content ourselves with an order-4 square, a literal equivalent of the last square we examined. In Figure 4, as promised, orthogonals and diagonals share the same set of 39 letters. An anagrammatic, alphamagic square will also survive, following translation into French. Or Swedish. Or Transylvanian. Or. . . ,

This is now what I mean by fool's gold: easily constructible greco-latin-based, higher-order alphamagic squares (or cubes, or whatever) exhibiting magical properties beyond the wildest fantasies of logomania. Seemingly marvelous trinkets that are none-

The Ancient Northumbrian *Li shu*

Five	: Twenty-two	: Eighteen
Twenty-eight	: Fifteen	: Two
Twelve	: Eight	: Twenty-five

5	22	18
28	15	2
12	8	25

The Ancient Chinese *Lo shu*

4	9	2
3	5	7
8	1	6

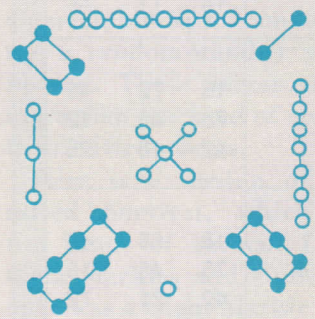


Figure 1

Diagonal 4×4 Greco-Latin Square

A+a	*B+b*	*C+c*	*D+d*
C+d	*D+c*	*A+b*	*B+a*
D+b	*C+a*	*B+d*	*A+c*
B+c	*A+d*	*D+a*	*C+b*

Figure 2

Greco-Latin-Based Magic Square

26	37	48	59
49	58	27	36
57	46	39	28
38	29	56	47

Figure 3

Literal Version of Figure 3

Twenty-six	*Thirty-seven*	*Forty-eight*	*Fifty-nine*
Forty-nine	*Fifty-eight*	*Twenty-seven*	*Thirty-six*
Fifty-seven	*Forty-six*	*Thirty-nine*	*Twenty-eight*
Thirty-eight	*Twenty-nine*	*Fifty-six*	*Forty-seven*

Figure 4

Alphamagic Squares Nos. 1–10

Index Numbers	Alphamagic Squares		
No. 1 (the *Li shu*)	5	22	18
	28	15	2
	12	8	25
No. 2	8	19	18
	25	15	5
	12	11	22
No. 3	15	72	48
	78	45	12
	42	18	75
No. 4	18	69	48
	75	45	15
	42	21	72
No. 5	21	66	48
	72	45	18
	42	24	69
No. 6	4	101	57
	107	54	1
	51	7	104
No. 7	44	61	57
	67	54	41
	51	47	64
No. 8	5	102	58
	108	55	2
	52	8	105
No. 9	45	62	58
	68	55	42
	52	48	65
No. 10	46	78	101
	130	75	20
	49	72	104

Figure 5. **The first ten English alphamagic squares of order 3, with their index numbers.**

theless just so many gewgaws, so much logological junk. And the reason resembles that in the case of the 3×3 German squares: limpidity robs them of interest; even casual inspection soon exposes their cheap reliance on the pantographic coupling between cardinals above 19 and the numbers of letters in their names. Worse still, in reality they are nothing more than tediously redundant diagonal greco-latin squares whose single identifiers have been expanded into words. All the "magic" they possess is entirely due to this underlying pattern, which guarantees (multi-level) uniformity of row, column, and diagonal content, no matter *what* (composite) entities replace its variables. Substituting chemical compounds for the latter, for instance, the distribution of chemical *elements* resulting would inevitably be magic as well. Greco-latins are of serious mathematical interest in their own right, to be sure, but in the guise of alphamagic squares they are only worthless imitations of the precious metal sought. We (and our friends) will need to be wary of these ironic pyrites.

Understand that the defect of such squares lies not exactly in their greco-latin morphology per se, but in their failure to disguise, to cover the traces of that foundation. A simple card-trick that defies explanation will continue to excite wonder as long as its mechanism remains invisible. The ingredient of concealment, of presenting a remarkable effect without giving away how it is achieved, is a sine qua non in any manifestation of "magic": magic minus mystery means "mundane."

Consider the *Li shu*, for instance, which, like any 3×3 magic square, is itself an instantiation of an order-3 greco-latin square—a characteristic that emerges quite clearly on checking the pattern of 1s, 2s, 5s, and 8s in its numerical representation (Figure 5, No. 1). But traces of that underlying structure are far from evident in the real or literal *Li shu*, a circumstance due to the usage of *twelve* instead of "ten-two," *fifteen* instead of "ten-five," and *eighteen* rather than "ten-eight"—usage, be it noted, that nevertheless preserves the word length of these more rational alternatives. Here then, in contrast to the transparency of the order-4 square above, we witness linguistic accident at work in the service of subterfuge, in helping to camouflage the tell-tale pattern that discloses its formative principle. Therein, in part, resides the power of the square, its claim to be Alpha-magic—in the sense of ranking first.

Bear in mind, incidentally, that—unlike the case in higher orders—the nonexistence of any 3×3 *diagonal* greco-latin square means that even the most conspicuous of order-3 alphamagics (No. 6 is the first) is always something more than a mere substitution of number-words for identifiers. Cardinals occurring along diagonals can never be just an alternative ordering of those composing every row and column; $a = (b+c)/2$ and $C = (A+B)/2$, recall. [These are the extra conditions to be satisfied in using the order-3 greco-latin square as a template for constructing a 3×3 magic square.] Comparison of literal versions of squares in the figure is instructive here; note that in terms of concealment, No. 2 betters No. 1. With our minds now alerted to these lesser greco-latin alloys, we return to the search for logologist's gold.

Following the successful approach used in deriving order-3 alphamagics, a good plan now would seem to be examination of the general formula for magic squares of order 4. This we shall do, but first let us glance at an intriguing possibility that is bound to suggest itself to anybody familiar with traditional magic-square theory.

Normal Squares

Sometime before 1675 (the date of his death), a French ecclesiastic, Bernard Frénicle de Bessy, first established that there are 880 distinct normal magic squares of order 4, excluding rotations and reflections. By *normal* is intended squares using the natural consecutive series 1, 2, 3, . . . , 16. A complete listing of the 880 was first published in 1693; ever since, they have attracted close attention, forming the subject of endless deliberations. The list can be found, for instance, in Benson and Jacoby's *New Recreations With Magic Squares* (Dover, 1976). Now, could it be that one of these traditional gems might prove to be alphamagic too?

Normal squares use the consecutive series 1, 2, 3, . . . , 16. Could one of these traditional gems prove to be alphamagic?

Note that this question, natural enough for order 4 (and higher) does not arise with order 3, for which there exists only one normal square—clearly nonalphamagic—the *Lo shu*. But how is it to be answered? At first sight the problem presents no insuperable difficulty since, in the last resort, a program could be written to generate and test every square in turn, a feasible if artless approach. However, the same question will reappear with order 5, for which Richard Shroeppel showed in 1973 that there are exactly 275,305,224 normal squares (again, excluding rotations and reflections), a figure for all practical purposes ruling out the brute-force method, on a personal computer at least. (For further details of Shroeppel's work, see Martin Gardner's excellent account in *Scientific American*, January 1976.) How then are we to determine whether one or more of these is alphamagic?

Surprisingly, an absurdly simple solution is to hand. Taking order 4 to begin with, notice that in any normal alphamagic square the words *one*, *two*, *three*, . . . , *sixteen* would appear. The total number of letters involved is thus 3 + 3 + 5 + . . . + 7 = 81. Hence the magic constant in the logorithm square, the number of letters occurring in all four rows (and all four columns), must equal one-fourth of this total. [Remember, the *logorithm* of a number is the number of letters in its written name.] But 81 is not divisible by 4. Therefore there are no normal alphamagic squares of order 4!

And what of the higher orders? What, in particular, is the lowest order N to fulfil the necessary (but not yet sufficient) condition

$$\left(\sum_{n=1}^{n=N^2} \log_e n\right) \bmod N = 0\ ?$$

Alas, not one of those 275,305,224 squares of order 5 could be alphamagic. Nor, indeed, will any of the unknown but assuredly astronomical number of order-6 squares answer. The astonishing fact is that we have to go up to order 14 (logorithm square magic constant = 189) before encountering an undisqualified candidate! And there yet remains the little matter of trying to identify an actual 14×14 normal alphamagic square.

The chance of success in seeking for such a monster seems remote in the extreme. Nevertheless, the problem is there and, conceivably, closer attention by intrepid programmers may discover means for delimiting the search so as to bring it within the scope of practical computer investigation.

The dispiriting result thus arrived at applies only to English squares, of course. Perhaps other languages will admit of lower-order solutions, an unexamined possibility some readers may like to explore—a more encouraging prospect for research, certainly, than the problem proposed above. What language, one wonders, will turn out to provide the lowest-order normal alphamagic square? Alternatively, how about near-normal squares using consecutive numbers, or even just arithmetic series other than $1-N^2$? Here are nice opportunities for chalking up some exotic "firsts" in computational logology. In any case, besides disposing of a seductive contingency, this digression has furnished a good example of how even seemingly sticky problems in the alphamagic sphere can unexpectedly yield to a cunning mixture of simple arithmetic and logo-logic. We shall now have need of all the cunning we can muster; it is time to turn to the ticklish problems indicated at the outset.

Formulae for Order 4

Reverting now to our original course, the obstacles to producing 4×4 alphamagic squares become clear on examining the general formula for magic squares of order-4 (see Figure 6). The method of construction here used, first described by J. Chernick in 1938 (*American Mathematical Monthly*, Volume 45, pages 172–5) and applicable to squares of any order, is simple and easy to follow. Starting at top left and filling in crosswise and downwards, cells in the top row are assigned independent variables p, q, r, and s; the magic constant C thus becoming $(p+q+r+s)$. t, u, and v follow in the next row, but if this is to total C, its final cell must contain $(C-t-u-v)$. Similarly, with w entered next, the lowest cell in the left-hand column becomes $(C-p-t-w)$, following which its

General Formula for Order 4

p	q	r	s
t	u	v	$p+q+r+s$ $-t-u-v$
w	$p+t+w$ $-s-v$	$q+r+2s$ $-t-u-w$	$u+v$ $-w$
$q+r+s$ $-t-w$	$r+2s+v$ $-t-u-w$	$p+t+u+w$ $-r-s-v$	$t+w$ $-s$

Figure 6

Greco-Latin-Based 4×4 Alphamagic Square

Eighteen (8)	*Twelve* (6)	*Twenty-three* (11)	*Five* (4)
Three (5)	*Twenty-five* (10)	*Nineteen* (8)	*Eleven* (6)
Sixteen (7)	*Thirteen* (8)	*One* (3)	*Twenty-eight* (11)
Twenty-one (9)	*Eight* (5)	*Fifteen* (7)	*Fourteen* (8)

Figure 7. **An English alphamagic square of order 4. Entries are transposable to form no fewer than 144 different alphamagics, every one of them exhibiting the 24 constellations here discoverable with magic constants 58;29.**

A Minimal Formula for Order 4

$A+a$	$B+b$	$C+c$	D
$C-x$	$D+c$	$A+b$	$B+a+x$
$D+b+x$	$C+a$	B	$A+c-x$
$B+c$	A	$D+a$	$C+b$

Figure 8

immediate diagonal neighbor can then be calculated. And so on. At a later stage, a further variable x must be introduced, only to be replaced by a compound expression later. (Try it; one Northumbrian proverb is worth a hundred of the Chinese variety.)

Looking over the formula, a few features common to all 4×4 magic squares quickly emerge: the four corner cells, the four center cells, the four inside cells of the outer rows, and the four inside cells of the outer columns all total to the magic constant $(p+q+r+s)$. Beyond these simple observations, however, there is little to be added. With order 3 we were able to point to cell groups comprising arithmetic triples, relying on this characteristic to narrow the area searched, but no such restriction is imposed on numbers appearing in order-4 (or larger) squares. On the contrary, in looking at cells containing single independent variables, we find that as many as half the entries may be numbers selected entirely at random.

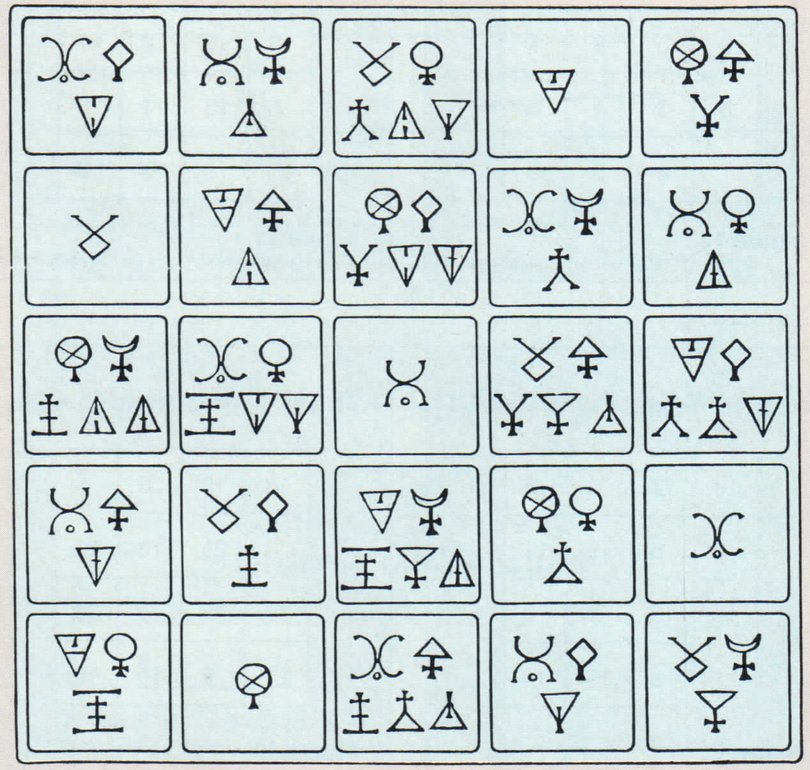
It is precisely this freedom, this absence of stricture in the choice of elements, that makes for the greatest difficulty in devising an alphamagic-divining algorithm. For without some further qualification regarding the properties of candidate entries (considered either individually or in relation to each other), the range of possible cases to be examined is simply boundless. And lacking criteria to limit the sweep of our search, is there any more reason to start looking in one direction than in another? Leaving aside number-crunching on a juggernaut scale, I for one have been unable to come up with a workable scheme for a computer program able to sift for solutions systematically, in any way analogous to the successful method evolved for order 3. Of course, others may yet succeed where the author has failed.

Defeated, then, in attempting to comb methodically for larger squares in general, as well as in trying to derive an example using the restricted set of numbers 1–16, the problem of how to produce even a single nontrivial 4×4 alphamagic square of any kind soon formed the focus of attention. Readers may judge of the eventual success of this mission from Figure 7, less a nugget retrieved from the ground than a product of patient alchemy. Appropriately, the Philosopher's Stone or essential catalyst necessary to this synthesis was revealed in a magical formula (see Figure 8).

Readers unfamiliar with magic-square material may be unaware that general formulae can appear in a variety of forms. Figure 8, for instance, is algebraically synonymous with Figure 6, the latter-named being a more redundant expression of exactly the same information. In fact, Figure 8, previously unpublished, is an example of what I term a *minimal* matrix; that is, one in which each of the eight necessary independent variables appears no more than four times in the square, the least possible number. (For a similar formula for order 5, see D. E. Knuth and L. Sallows, Problem 1296 in the *Journal of Recreational Mathematics*, Vol. 16, No. 2, 1983–4.) (*JRM*, by the way, should not be confused with a rival publication, the sadly maligned *Journal of Rejected Manuscripts*.) The derivation of minimal matrices, incidentally, is a small chapter of magic-square theory in itself; see Figure 9. But how can the minimal formula be of use in creating alphamagic squares?

For an answer to this, examine the placing of A, B, C, and D in Figure 8: a pattern comprising a diagonal latin square. The same is true of a, b, and c, although here the fourth expected identifier d is missing. Notice that no two cells are alike in content. Besides this, two positions are occupied by x and two by $-x$, an arrangement leaving the magic constant

Near-Minimal Magic-Square Formulae, Old and New



$A + c + y$	$a + y$	$C + x$	$B + b + x$
$C + b + y$	$B + x$	$A + a + y$	$c + x$
$B + a + x$	$C + c + y$	$b + x$	$A + y$
x	$A + b + x$	$B + c + y$	$C + a + y$

Figure 9. Above: **A diagram found in Agrippa von Nettesheim's *De Occulta Philosophia* (Noviomagus edition), published in Lyons, 1533. Cabalistic symbols replace *a, b, c*, . . . , in representing independent variables. Inverted versions of the same signs correspond to $-a$, $-b$, $-c$, . . . (compare the lower symbols among the cells). Astrological significance was attached to such magic diagrams. Rewriting the diagram in familiar notation produces what we recognize as an algebraic formula. Further elementary algebra is needed to show that the formula is a universal generalization, embracing all possible 5×5 magic squares. Even so, the number of variable occurrences is 81—12 more than the 69 required in a minimal formula for order 5.**

Below: **A modern near-miss at a 4×4 minimal formula due to John Horton Conway, a mathematical deity to whom we owe the gift of *Life* (taken from a letter to H. S. M. Coxeter, dated 7 March 1957). The number of variable occurrences in this square is 40—8 more than the 32 used in Figure 8, the minimal formula. To see that at least 32 must appear, note that Chernick's method (Figure 6) proves that 8 *independent* variables are involved. A little thought shows that, however arranged, each of these will have to appear 4 times in the square if it is to be magic.**

Primary Latin Square

8	1	3	5
3	5	8	1
5	3	1	8
1	8	5	3

Figure 10

First Transform (a=10)

18	1	3	5
3	5	8	11
5	13	1	8
1	8	15	3

Figure 11

Second Transform (b=11)

18	12	3	5
3	5	19	11
16	13	1	8
1	8	15	14

Figure 12

Third Transform (c=20)

18	12	23	5
3	25	19	11
16	13	1	28
21	8	15	14

Figure 13

Alphamagic Reshuffle of Figure 13

12	23	8	15
18	5	21	14
25	19	13	1
3	11	16	28

Figure 14

$(A+B+C+D+a+b+c)$ everywhere unaffected. Thus, carefully considered, the message contained in the matrix is that every 4×4 (alpha)magic square is decomposable into a diagonal greco-latin square (in which it will turn out that element $d=0$, and hence need not appear)—distorted slightly, as it were, by a quantity corresponding to the simple zero-totalling pattern of x's. Then, recalling a comparable analysis of the *Li shu,* a *significant* or *interesting* order-4 alphamagic square could only be one in which this predominantly greco-latin substrate had been largely obscured. And properly assimilated, the effect of this insight is to suggest an entirely novel approach to the construction of such squares: calculated exploitation of linguistic accident with a view to *transforming* a trivial, easy-to-construct square.

It is here that pencil and paper can often supplement keyboard and screen, as the process of creation partly involves experimental tinkering and serendipity, a factor notoriously intransigent to algorithmic encapsulation. Detailed elucidation of the technique is therefore to some extent an exercise in rationalized reconstruction. The procedure can be illustrated, though, through a reduplication of Figure 7, itself retraceable to a primary diagonal latin (and thus alphamagic) square, shown as Figure 10.

Here, in effect, we have an instantiation of the general formula in which $A=8$, $B=1$, $C=3$, $D=5$, while $a=b=c=x=0$. That the numbers 1, 3, 5, and 8 have not been selected without careful premeditation is seen from the following relations:

$$\log 1 + \log 10 = \log 11$$
$$\log 3 + \log 10 = \log 13$$
$$\log 5 + \log 10 = \log 15$$
$$\log 8 + \log 10 = \log 18$$

Noting that *eleven, thirteen,* . . . can be substituted for *ten+one, ten+three,* . . . without change to the logorithms, setting $a=10$ in the formula produces a slightly less trivial alphamagic square, Figure 11. We can do better than this, however, as the linguistic coincidences associated with 1, 3, 5, 8 (arrived at through previous random experimentation) have not yet been exhausted:

$$\log 1 + \log 11 = \log 12 + 3$$
$$\log 3 + \log 11 = \log 14 + 3$$
$$\log 5 + \log 11 = \log 16 + 3$$
$$\log 8 + \log 11 = \log 19 + 3$$

Here again, setting $b=11$ in the formula, the constant difference between new entries and old at both numerical and logorithmic levels results in no adverse effects on alphamagic properties, as shown in Figure 12.

Only one more such trick and we shall have a matrix using sixteen distinct numbers. Falling back on an obvious standby, setting $c=20$, our first nontrivial alphamagic square of order 4 will then be complete (Figure 13).

Since x in the formula is still equal to zero, this final matrix (Figure 7) is in fact patterned on a pure greco-latin square. As a consequence, it enjoys certain extra (alpha)magic properties special to that structure. In particular, the four cells in each quadrant and the four corner cells of each 3×3 subsquare also total to the magic constants 58;29. By happy coincidence, 58 is exactly twice 29. Moreover, cyclic permutations and other transpositions of its elements mean that the 16 cardinals here employed can be rearranged into no less than 144 distinct alphamagic squares (not counting rotations and reflections), every one of them displaying the 24 alphamagic constellations displayed by Figure 7. This ability to reshuffle can sometimes be used to manipulate elements into strategic positions. In Figure 14, for example, 1, 14, 18, and 25 have been maneuvered into the four cells occupied by x's in the general formula.

Having already established (again, by trial and error) that

$$\begin{aligned} \log 1 &= (\log 0) - 1 \\ \log 14 &= (\log 15) + 1 \\ \log 18 &= (\log 17) - 1 \\ \log 25 &= (\log 26) + 1, \end{aligned}$$

setting $x=1$ leads to a square in which 0, 15, 17, 26 replace 1, 14, 18, 25—a transformation marred, however, by the double occurrence of 15 in the resulting matrix. Patience, though, discovers a way over the difficulty by adding 19 to the number represented by a in the formula, to produce a nontrivial alphamagic square no longer founded on a simple greco-latin square. The appearance of "zero" strikes me as an especially felicitous touch (see Figure 15).

Enough said about this somewhat makeshift method of construction, whose introduction has admittedly been very much a stopgap measure, primarily designed to smooth over an embarrassing semicompletion. Admittedly, systematic computer searches might go a long way toward displacing

A Non-Greco-Latin-Based Alphamagic Square

Thirty-one (9)	*Twenty-three* (11)	*Eight* (5)	*Fifteen* (7)
Seventeen (9)	*Five* (4)	*Twenty-one* (9)	*Thirty-four* (10)
Twenty-six (9)	*Thirty-eight* (11)	*Thirteen* (8)	*Zero* (4)
Three (5)	*Eleven* (6)	*Thirty-five* (10)	*Twenty-eight* (11)

Figure 15

General Formula for Order-3 Magic Cube

a	$-a-b$	b
$-a-c$	$a+b+c+d$	$-b-d$
c	$-c-d$	d

$-a+d$	$a+b-c-d$	$-b+c$
$a-b+c-d$	0	$-a+b-c+d$
$b-c$	$-a-b+c+d$	$a-d$

$-d$	$c+d$	$-c$
$b+d$	$-a-b-c-d$	$a+c$
$-b$	$a+b$	$-a$

Figure 16. **General formula for a magic cube (zero-sum form; add *k* to every cell for true generalization).**

Concentric Alphamagic Square of Order 5

Fifty-nine (9)	*Eighty-nine* (10)	*Seventeen* (9)	*Forty-four* (9)	*Sixty-one* (8)
Sixty-seven (10)	*Four* (4)	*One hundred one* (13)	*Fifty-seven* (10)	*Forty-one* (8)
Fifteen (7)	*One hundred seven* (15)	*Fifty-four* (9)	*One* (3)	*Ninety-three* (11)
Eighty-two (9)	*Fifty-one* (8)	*Seven* (5)	*One hundred four* (14)	*Twenty-six* (9)
Forty-seven (10)	*Nineteen* (8)	*Ninety-one* (9)	*Sixty-four* (9)	*Forty-nine* (9)

Figure 17

General Formula for Concentric Alphamagic Square of Order 5

$a+e$	$a+i$	$a-e-f-h-i$	$a+h$	$a+f$
$a+g$	$a+b$	$a-b-c$	$a+c$	$a-g$
$a-e+f-g-j$	$a-b+c$	a	$a+b-c$	$a+e-f+g+j$
$a+j$	$a-c$	$a+b+c$	$a-b$	$a-j$
$a-f$	$a-i$	$a+e+f+h+i$	$a-h$	$a-e$

Figure 18

serendipity, although the factor not to be underrated here is the problem of stipulating exactly what properties are being sought. In any case, the need for a better algorithm becomes highlighted when we acknowledge the impossibility of assigning index numbers to squares formed in this fashion.* Here we have a problem pleading for solution by computer, and I am curious to learn how better-equipped programmers will rise to the challenge, as doubtless some will. (Can someone produce English 4×4 Alphamagic Square No. 1? Remember, it need not be *attractive*, logologically speaking. And can anyone identify the index numbers of Figures 7 and 15?)

In the meantime, the manually-aided transmutation of greco-latin alloy into logological gold is a viable alternative, and I hope some readers will be encouraged to absorb the details above and go on to construct new squares of their own. For any who enjoy a puzzle, as well as the chance of making novel finds, it is an absorbing pastime, in some ways more akin to a skill than a science. And the field is not limited to order 4, or even to alphamagic *squares.* Figure 16 gives a new general formula for a 3×3×3 magic *cube*, of which even one alphamagic example has yet to be discovered.

Lastly, in Figure 17, I beg to present a final specimen of the alphamagic art, the fruit of pensive nights and laborious days: a *concentric* alphamagic square of order 5, in which the outer layer of cells can be peeled off to leave a central alphamagic square of order 3 (No. 6). The formula for such a square is shown in Figure 18; let any who would improve upon this by all means try a hand.

*Part I described a simple ranking system whereby every alphamagic is associated with a unique index number.

Conclusion

This has been a relatively brief reconnaissance in an unfrequented border country between the Mountains of Mathematics and the Lowlands of Logology, a hitherto unsuspected realm brought to light through *The Origin of Tree Worship*. (For an account of travels in some adjoining regions, however, see the last three items on the reference list.) One unanticipated consequence of our alphamagic journey has been to discover how comparatively little is actually demanded of an arrangement of numbers in qualifying as an ordinary or, if you will, *beta*-magic square.

Some may feel that here is something that Schroeppel's finding of 275-million-odd normal 5×5 squares should have made plain long since. Possibly so. Notwithstanding, innumerable publications in the field attest to a widespread, irrational susceptibility to traditional magic squares—a seemingly unflagging appetite for the cataloging of new specimens, no matter how inexhaustible in supply, how underwhelming and unworthy of attention they turn out to be on sober assessment. Many contributors avoid the worst excesses of this tendency, it is true, yet a surfeit of exclamation marks is almost a hallmark of the magic-square literature. It would be nice to think that the reference here was to factorials, the notation "$N!$" standing for $1\times2\times3\times...\times N$ (a common enough occurrence, as it happens, in formulae relating to enumerations). The truth of the matter, however, is less prodigious, the apparent superfluity resulting only from the too-often-encountered gasp of ". . . *magic!!!!*" expressed by authors moved to raptures over yet another find.

The advent of alphamagic squares promises a breath of fresh air in this respect, as their simultaneous compliance with magic requirements at two separate levels, outclasses the familiar prototypes and gives pause for reassessment in the field. And their unimposing, even whimsical exterior casts incidental light on the supposedly *mathematical* nature of ordinary magic squares—a view, I would suggest, as mistaken as it is pervasive.

The widely-held apprehension of magic squares as intrinsically mathematical objects is really a false impression encouraged by the sight of numbers in matrices. The genuine mathematical *problems* involved in their construction and enumeration reinforce this image. But rich as they are in mathematical connotations, the structures themselves—the completed squares—are not merely trivial but actually vacuous in any true mathematical sense: they impart no mathematical information, identify no mathematical relationships, possess no mathematical significance.

More essentially, the fascination they command, the interest they provoke, lies in the intriguing, counterintuitive quality of *co-incidence* they embody. Herein only resides the "magic"—a quality evinced to a greater degree by their alphamagic successors. It is their concomitant satisfaction of independent constraints that calls forth wonder, the role of the numbers as such being less central than first sight supposes: at root, these are simply a vehicle for the expression of the magical effect. Gardner's anagram, "*Eleven + two = twelve + one*", for instance, also exploits numbers to produce a magic constant, but we hardly see that as "mathematical." Likewise, the matrix or square arrangement has nothing to do with real matrices, being only a catchy device for marking off certain subsets.

For all that, one cannot ignore an important aspect of these structures that truly partakes of a mathematical nature: their abstract or Platonic status, independent of empirical reality, reflecting an absolute truth beyond all qualification of time or space. Should it emerge that intelligent creatures on some far-flung planet possess magic squares, we may be sure that theirs will be the same as ours.

Can the same be said of alphamagic squares? To the extent that these include the former, yes. But what of the mutability of number-representations? "Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not." Such is the opinion of G. H. Hardy, the great English number theorist, in his tender testimonial *A Mathematician's Apology*. Yet without notations, names, languages to re-member them, how could there be any mathematical ideas? The duality of sign and signified is ineluctable. Doubtless the alphamagic squares of the Alpha Centaurians, say, will differ a trifle from terrestrial types, even as they remind us that the alphamagic *principle* remains transcendent.

Alphamagic squares comply with magic requirements at two separate levels, outclassing the familiar prototypes.

We come thus to the close of this preliminary investigation touched off by the rediscovery of the *Li shu*, itself no mere example of its kind, but the great archetype of alphamagic squares in all the tongues of the globe, as the results of this research have shown. Perhaps we should not be surprised at the English origin of the legend, the alphamagic principle evident-

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many of them, but he wanted to demonstrate what appears to be an arbitrary model of technological development. In particular, he wanted to prove the validity of his own pet 'law'—the suppression of radical potential. This law seems to be based not on how technology is handled, but on some socialistic paranoia regarding deliberate interference by the powerful elements of society in opposition to technological progress for the benefit of mankind. Consequently, Winston has written a very detailed history of computers (and three other parts of the information revolution) which is false because it works from the point of view of how things come out, and because it ignores, misstates, and gets wrong important parts of that history.

This is too bad, because Winston has some important things right, principally the appropriate ordering of scientific discovery and technological invention. Contrary to the usual assumptions, they do not have that order. They do not have any order. Sometimes one and sometimes the other is first. Sometimes they both are, and sometimes neither is, and often they cannot be distinguished.

It is too bad because it would be nice to have a model of technological development that would help us with the problem of how we too can do great things. But it is better to have no model than to have one that is wrong. It is better to say—the best thing we can now say—that technological development appears to be an unordered scrambling melee. It is all right to search for order in a melee, but it is wrong to insist that you have found such order when you have not.

Finally, it is too bad because, as I said at the outset, one does not expect the Harvard University Press to produce a book as flawed as this one in the year of the 350th anniversary of the founding of the college.

Alphamagic Squares
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ly finding its most perfect manifestation in the runes of Anglo-Saxon Northumbria. But whence came the magical formula? For, regretfully, the modern mind must reject the spirit of the yew tree as a primitive if colorful superstition. And yet, if we discount the supernatural agency, of the tree spirit, who then was the human author? Alas, no name comes down to us from the veiled centuries of prehistory.

A Druid he was, no doubt, one high in the standing of King Mī, perhaps; a master of abacus and alphabet who commanded leisure for the pursuit of learning in the service of religious ritual and magic. Had this runemaster so chosen, can we doubt that he would have found means to pass on his name to posterity, so cunning a mind, a mixture of Merlin and Mycroft? Mayhap it was natural modesty. Or was he yet one who understood that where there is no mystery there can be no real magic?

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